Calorimeter-less gamma-ray telescopes: Optimal measurement of charged particle momentum from multiple scattering by Bayesian analysis of Kalman filtering innovations

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http://llr.in2p3.fr/~dbernard/polar/harpo-t-p.html
W-slab Converter $\gamma \rightarrow e^+ e^-$ Telescopes

Fermi LAT Performance, Pass 8 Release 2 Version 6

"Fermi-LAT below 100 MeV (Pass8 data)", J. McEnery,
"e-ASTROGAM workshop, Padova 2017"
High-Angular Resolution $\gamma \rightarrow e^+e^-$ Telescope Projects

- Lower $Z$ and/or lower density, higher spatial resolution
  
  W-less, Si-stack detectors  
  AMEGO, e-ASTROGAM  
  $1.3^\circ \oplus 100$ MeV  
  A. De Angelis et al., Exp. Astr. 44 (2017) 25

  Emulsions  
  GRAINE  
  $1^\circ \oplus 100$ MeV  
  S. Takahashi et al., PTEP 2015 (2015) 043H01

  Gas time projection chamber (TPC)  
  HARPO  
  $0.4^\circ \oplus 100$ MeV  
  D. Bernard, NIM. A 701 (2013) 225

Polarimetry with $\gamma \rightarrow e^+e^-$:

- Lower effective area - per - tracker—converter mass

  $\Rightarrow$ Larger volume?  
  $\Rightarrow$ Calorimeter mass budget?  
  $\Rightarrow$ $\gamma$-energy measurement?

  $\Rightarrow$ charged-track momentum measurement?
Multiple Scattering & Track Momentum Measurement

Theory of the Scattering of Fast-Charged Particles (G. Molière):

- Fast charged particles deflected by detector ion and electron electric field

- These small deflections average to a Gaussian angle distribution
  II. Plural And Multiple Scattering, Z.Naturforsch. A3 (1948) 78

In modern form: C. Patrignani et al. (Particle Data Group), Chin. Phys. C, 40, 100001 (2016)

\[
\theta_{RMS} = \frac{p_0}{\beta c p} \sqrt{\frac{x}{X_0}} \left(1 + \epsilon \log \frac{x}{X_0}\right)
\]

- \( p \) track momentum, \( p_0 = 13.6 \text{ MeV}/c \) “multiple scattering constant”
- \( x \) path length and \( X_0 \) scatterer radiation length (cm)

- Measuring the statistics of these deflections yields a momentum measurement

Momentum Measurement: Molière Method

Widely used for decades:

- Emulsion trackers
-Muon detector
  - Sampling detectors (instrumented Fe or Pb layer stacks)
  - lAr (liquid Argon) kilo-ton time projection chambers (TPC) for neutrino studies.

- ... 


- Track segment into tracklets, tracklet angle measured, deflection angle between tracklets $n$ and $n + 1$ computed.

- How decide the segment length?
Thin Detectors: Segment Length Optimisation

- Relative momentum RMS resolution:
  \[ \frac{\sigma_p}{p} = \frac{1}{\sqrt{2L}} \left[ \Delta^{1/2} + \frac{p^2 \sigma^2 X_0}{\Delta^{5/2} p_0^2} \right], \]

- \( \Delta \) segment length, \( \sigma \) single track measurement spatial precision, \( L \) total track length. Minimum for:
  \[ \Delta = \left[ \frac{5p^2 \sigma^2 X_0}{p_0^2} \right]^{1/3}. \]

- Optimal segmentation depends on momentum to be measured!

- The value of \( \sigma_p/p \) for that optimal set is:
  \[ \frac{\sigma_p}{p} = \frac{C}{\sqrt{2L}} \left[ \frac{p}{p_0} \right]^{1/3} \left[ \sigma^2 X_0 \right]^{1/6}, \quad C \equiv 5^{1/6} + 5^{-5/6} \approx 1.57. \]

Relative track momentum resolution for optimal sampling \( \Delta \) as a function of track momentum. (Argon TPC)

D. Bernard, NIM. A 701 (2013) 225
Towards an Optimal Method?

- How extract optimally all the multiple scattering information present in the track given the single track measurement spatial precision?
- Kalman-filter tracking yields an optimal estimate of the track parameters (e.g., lateral position and track angle at $z = 0$)
- Optimal treatment of multiple scattering and single track measurement spatial precision (Gaussian approximation)
- Kalman filter tracking yields an optimal estimate of the track parameters at point $n + 1$ from an optimal combination of
  - an optimal estimate of the track parameters at point $n$
  - the measurement at point $n + 1$


- The update of the track parameters and of their covariance matrix involves
  - the track covariance matrix at $n$
  - the measurement covariance matrix (spatial resolution)
  - the process noise covariance matrix (multiple scattering, hence track momentum!)
Optimal Method

- $z_n$, measurements

- $x_{n-1}^n$, prediction (at $n$, from the analysis of the points from 0 up to $n - 1$)

- $\nu_n \equiv z_n - x_{n-1}^n$, innovations

  $\nu_n$ are Gaussian distributed, $\mathcal{N}(0, S_n(s))$

  $S_n(s)$, covariance matrix is computed while the Kalman filter, given $s$, is proceeding along the track

  $s \equiv \left(\frac{p_0}{p}\right)^2 \frac{\Delta x}{l X_0}$ is the average multiple-scattering angle variance per unit track length,

  $\theta^2 = s \times l$.  

  $l$ longitudinal sampling,  $\Delta x$ scatterer thickness.  Homogeneous detector:  $l = \Delta x$.

- $p_n(s)$ probability to observe such a track up to $n$, given $s$

- It can be shown that $p_n(s) \propto \prod_i \mathcal{N}(\nu_i(s), 0, S_i(s))$

Optimal Method: Results

$p_N(s)$ distribution for a 50 MeV/$c$ track in a silicon detector

$X_0 \quad 9.4 \quad \text{cm}$

$l \quad 1.0 \quad \text{cm}$

$\Delta x \quad 0.0500 \quad \text{cm}$

$\sigma \quad 0.0070 \quad \text{cm}$

$N \quad 56$

Obtain $s$ that maximises $p_N(s)$ and $p = p_0 \sqrt{\frac{\Delta x}{lX_0s}}$

Performances

Silicon detector.

Unbiased, usable up to a couple of GeV/c


$10^4$ evts / MC sample.
Conclusion

Magnetic-field-free trackers:

- Molière method (1955): measures charged-track momentum from multiple measurements of multiple-scattering-induced deflections

- Kalman-filter track fit (Frühwirth, 1987): yields optimal unbiased charged-track parameters when the momentum is known.

- A Bayesian analysis of the filtering innovations of $s$-indexed Kalman filters yields an optimal, unbiased, estimate of the momentum (Frosini & Bernard 2017)
  - and of the other track parameters, BTW.

- Caveat: a number of approximations:
  - no energy loss in detector
  - no radiation
  - Gaussian multiple-scattering angle distribution and space resolution
  - ...
Parametrisation of the relative momentum resolution

A good representation of these data

\[ \frac{\sigma_p}{p} \approx \frac{1}{\sqrt{2N}} \sqrt[4]{1 + 256 \left( \frac{p}{p_0} \right)^{4/3} \left( \frac{\sigma^2 X_0}{N \Delta x l^2} \right)^{2/3}} \]

Low-momentum asymptote

\[ \frac{\sigma_p}{p} \approx \frac{1}{\sqrt{2N}} \]

High-momentum asymptote

\[ \frac{\sigma_p}{p} \approx \sqrt[4]{\frac{8}{N} \left( \frac{p}{p_0} \right)^{1/3} \left( \frac{\sigma^2 X_0}{N \Delta x l^2} \right)^{1/6}} \]

\( p_s \), the momentum above which \( \sigma_p/p \) starts to depart from the low momentum asymptote,

\[ p_s = p_0 \frac{1}{64} \left( \frac{N \Delta x l^2}{\sigma^2 X_0} \right)^{1/2} \]

\( p_\ell \), the momentum above which \( \sigma_p/p \) is larger than unity (meaningless measurement)

\[ p_\ell = p_0 \left( \frac{N}{8} \right)^{3/2} \left( \frac{N \Delta x l^2}{\sigma^2 X_0} \right)^{1/2} \]

Note that

\[ p_\ell = p_s (2N)^{3/2} \]

And

\[ \frac{\sigma_p}{p} \approx \frac{1}{\sqrt{2N}} \sqrt[4]{1 + \left( \frac{p}{p_s} \right)^{4/3}} \]

Parametrisation of the relative momentum resolution

High-momentum asymptote

\[
\frac{\sigma_p}{p} \approx \sqrt{\frac{8}{N}} \left( \frac{p}{p_0} \right)^{1/3} \left( \frac{\sigma^2 X_0}{N \Delta x L^2} \right)^{1/6}
\]

\[
L = l \times N
\]

\[
\frac{\sigma_p}{p} \approx \sqrt{8} \left( \frac{p}{p_0} \right)^{1/3} \left( \frac{\sigma^2 X_0}{N^2 \Delta x L^2} \right)^{1/6} \approx \sqrt{8} \left( \frac{p}{p_0} \right)^{1/3} \left( \frac{\sigma}{NL} \right)^{1/3} \left( \frac{X_0}{\Delta x} \right)^{1/6}.
\]

- For a given wafer thickness, \( \Delta x \), and total detector thickness \( L \), improvement with larger \( N \) (smaller \( l \))
  - part of it by improving the precision of the position over a given segment length \( \frac{\sigma}{\sqrt{N}} \)
  - the rest of it, \( \left( \frac{1}{\sqrt{N}} \right)^{1/3} \), use of short scale multiple scattering information

- For homogeneous active targets, \( \Delta x = l \) (gas TPC telescopes, LAr TPC \( \nu \) detectors)

\[
\frac{\sigma_p}{p} \approx \sqrt{8} \left( \frac{p}{p_0} \right)^{1/3} \left( \frac{\sigma^2 X_0}{NL^3} \right)^{1/6}
\]

i.e., the same residual \( \left( \frac{1}{\sqrt{N}} \right)^{1/3} \), scaling